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On Second Kind Measures and Polynomials on the Unit Circle

LEONID GOLINSKII*

Mathematical Division, Institute for Low Temperature Physics and Engineering, 47 Lenin Avenue, Kharkov 310164, Ukraine

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Properties of second kind polynomials, and, in particular, conditions for second kind measures to be absolutely continuous are investigated. The asymptotic representation for second kind polynomials is obtained. Examples of generalized Jacobi weighted functions are considered.

1. INTRODUCTION

The theory of polynomials orthogonal on the unit circle was developed by G. Szegő, Ya. L. Geronimus, and G. Freud (see their monographs [Sz], [Ge1], [Fr] and surveys [Ge2], [Ge3], [N]). We first recall the definitions as well as some results from this theory.

Let $d\sigma$ be a finite positive Borel measure on the interval $[-\pi, \pi]$ with infinite support,

$$d\sigma(t) = \varphi(t)\, dt + d\sigma_s(t)$$

be its Lebesgue decomposition. We call the function $\varphi \in L^1[-\pi, \pi]$ a density function of the measure $d\sigma$. In what follows, L^p stands for $L^p[-\pi, \pi]$, $p \geqslant 1$, and

$$\|f\|_p = \left\{\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^p \, dt\right\}^{1/p}.$$

Throughout the paper we denote $\zeta = e^{it}$, where t is a real parameter. We write $c_k(d\sigma)$ for the moment sequence

$$c_k(d\sigma) = (2\pi)^{-1} \int_{-\pi}^{\pi} \zeta^k \, d\sigma(t), \qquad k = 0, 1, \ldots .$$

* E-mail address: golinskii%ilt.kharkov.ua@relay.ussr.eu.net.

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Given a measure $d\sigma$, there exists a uniquely determined system of orthonormal polynomials $\{\varphi_n\}_{n\geqslant 0}$ such that

$$(2\pi)^{-1}\int_{-\pi}^{\pi}\varphi_n(\zeta)\,\overline{\varphi_m(\zeta)}\,d\sigma(t)=\delta_{n,m},\qquad n,m=0,1,..., \tag{1.1}$$

and

$$\varphi_n(z)=\kappa_n z^n+\cdots,\qquad \kappa_n>0.$$

The Carathéodory function (C-function)

$$F(z)=\frac{1}{2\pi c_0}\int_{-\pi}^{\pi}S(t,z)\,d\sigma(t),\qquad c_0=c_0(d\sigma),\ S(t,z)=\frac{e^{it}+z}{e^{it}-z}, \tag{1.2}$$

is analytic in the unit disk $\mathbb{D}=\{z:|z|<1\}$ and $\operatorname{Re}F(z)>0$ for $z\in\mathbb{D}$. If $z=re^{i\theta}$ and

$$P(r,x)=\frac{1-r^2}{1-2r\cos x+r^2},\qquad Q(r,x)=\frac{2r\sin x}{1-2r\cos x+r^2},$$

then

$$S(t,z)=\frac{1+re^{i(\theta-t)}}{1-re^{i(\theta-t)}}=P(r,\theta-t)+iQ(r,\theta-t),$$

so that

$$\begin{aligned}F(re^{i\theta})&=\frac{1}{2\pi c_0}\int_{-\pi}^{\pi}P(r,\theta-t)\,d\sigma(t)+\frac{i}{2\pi c_0}\int_{-\pi}^{\pi}Q(r,\theta-t)\,d\sigma(t)\\&=u(r,\theta)+iv(r,\theta).\end{aligned} \tag{1.3}$$

For the C-function F

$$\lim_{r\to 1-0}\operatorname{Re}F(re^{i\theta})=\lim_{r\to 1-0}u(r,\theta)=\frac{\varphi(\theta)}{c_0(d\sigma)} \tag{1.4}$$

exists a.e. on $[-\pi,\pi]$, and the inversion formula is

$$\frac{\sigma(\theta_2+0)+\sigma(\theta_2)}{2}-\frac{\sigma(\theta_1+0)+\sigma(\theta_1)}{2}=c_0\lim_{r\to 1-0}\int_{\theta_1}^{\theta_2}\operatorname{Re}F(re^{it})\,dt, \tag{1.5}$$

where $\sigma(\theta)$ stands for $\sigma\{[-\pi,\theta)\}$ (with regard to the relations (1.4), (1.5) see, e.g., [Ge2, Sect. 11]). Note, also, that if the measure $d\sigma$ has a mass point t_0, then the value $\sigma(\{t_0\})$ can be calculated by means of the relation [Ge2, Sect. 15]

$$\sigma(\{t_0\}) = \pi c_0 \lim_{r \to 1-0} F(re^{it_0})(1-r). \tag{1.6}$$

For the measures $d\sigma$ in the Szegő class, that is, when

$$\int_{-\pi}^{\pi} \ln \sigma'(t)\, dt > -\infty, \tag{S}$$

the principal tool is the Szegő function

$$D(\sigma, z) = \exp\left\{(4\pi)^{-1} \int_{-\pi}^{\pi} \frac{\zeta + z}{\zeta - z} \ln \sigma'(t)\, dt\right\}.$$

As is known (see [Sz, Chap. 10], [Fr, Chap. 5]), $D(\sigma) \in H^2(\mathbb{D})$, $D(\sigma, 0) > 0$, and for the radial boundary values the relation

$$|D(\sigma, e^{it})|^2 = \sigma'(t)$$

holds a.e. with respect to Lebesgue measure.

In the present paper we investigate the behavior of the so-called second kind polynomials and measures (cf. [Ge1, Chap. 1], [Go1], [R]). The second kind polynomials are defined by $\psi_0(z) = 1/\sqrt{C_0}$,

$$\begin{aligned} \psi_n(z) &= \frac{1}{2\pi c_0} \int_{-\pi}^{\pi} S(t, z)[\varphi_n(e^{it}) - \varphi_n(z)]\, d\sigma(t) \\ &= \kappa_n z^n + \cdots, \qquad \psi_n(0) = -\varphi_n(0), \qquad n = 1, 2, \ldots. \end{aligned} \tag{1.7}$$

An important relation between the φ_n and ψ_n is given by

$$\varphi_n^*(z)\, \psi_n(z) + \psi_n^*(z)\, \varphi_n(z) = 2c_0^{-1} z^n, \qquad n = 0, 1, \ldots. \tag{1.8}$$

Here the *-transform of an nth degree polynomial P is defined by

$$P^*(z) = z^n \overline{P(1/\bar{z})},$$

where the conjugation refers to taking the complex conjugates of the coefficients of the polynomial P. The polynomials ψ_n are orthonormal with respect to the uniquely determined second kind measure $d\tau(t)$, that is,

$$(2\pi)^{-1} \int_{-\pi}^{\pi} \psi_n(\zeta)\, \overline{\psi_m(\zeta)}\, d\tau(t) = \delta_{n,m}, \qquad n, m = 0, 1, \ldots.$$

The moment sequences $c_k(d\sigma)$ and $c_k(d\tau)$ are related to each other by the identities

$$c_0(d\tau)=c_0(d\sigma), \qquad c_k(d\tau)=-c_k(d\sigma)-\frac{2}{c_0(d\sigma)}\sum_{j=1}^{k-1} c_j(d\sigma)\, c_{k-j}(d\tau)$$

(cf. [Ge2, Sect. 7]), whence follows the equality for the corresponding C-function G,

$$G(z)=\frac{1}{2\pi c_0}\int_{-\pi}^{\pi} S(t,z)\, d\tau(t)=\{F(z)\}^{-1}, \tag{1.9}$$

where $c_0=c_0(d\sigma)=c_0(d\tau)$ (cf. [Gol, p. 130]). Therefore (see (1.4))

$$\lim_{r\to 1-0} \operatorname{Re} G(re^{i\theta})=\lim_{r\to 1-0} \operatorname{Re}\{u(r,\theta)+iv(r,\theta)\}^{-1}$$

$$=\lim_{r\to 1-0}\frac{u(r,\theta)}{u^2(r,\theta)+v^2(r,\theta)}$$

and hence a.e. on $[-\pi,\pi]$

$$\tau'(\theta)=c_0^2\,\frac{\varphi(\theta)}{\varphi^2(\theta)+\tilde{\varphi}_\sigma^2(\theta)}, \tag{1.10}$$

where

$$\tilde{\varphi}_\sigma(\theta)=\lim_{r\to 1-0}\frac{1}{2\pi c_0}\int_{-\pi}^{\pi} Q(r,\theta-t)\, d\sigma(t) \tag{1.11}$$

(cf. [Gol, Lemma 1], [R, p. 107]). The function $\tilde{\varphi}_\sigma$ is called a conjugate function to the measure $d\sigma$ (if $d\sigma$ is absolutely continuous, then $\tilde{\varphi}_\sigma$ is just an ordinary conjugate function to $\varphi=\sigma'$). It is well known (cf. [Gel, Chap. 8, Theorem 8.2]) that

$$d\sigma\in S \Leftrightarrow \sum_{k=0}^{\infty}\left|\frac{\varphi_k(0)}{\kappa_k}\right|^2<\infty \Leftrightarrow \sum_{k=0}^{\infty}\left|\frac{\psi_k(0)}{\kappa_k}\right|^2<\infty \Leftrightarrow d\tau\in S.$$

We write $d\sigma\in AC[a,b]$, where $-\pi\leqslant a<b\leqslant\pi$, if the measure $d\sigma$ is absolutely continuous on $[a,b]$, and $d\sigma\in AC(a,b)$, if for each interval $[a_1,b_1]\subset(a,b)$ the measure $d\sigma$ is absolutely continuous on $[a_1,b_1]$ ($d\sigma\in AC$ means that the measure $d\sigma$ is absolutely continuous on the whole interval $[-\pi,\pi]$). As usual, we denote by $C_{2\pi}$ the class of continuous 2π-periodic functions on the real line. We write $\omega(\delta,f)$ for the modulus of continuity of a function $f\in C_{2\pi}$, that is,

$$\omega(\delta,f)=\sup_{|x-y|\leqslant\delta}|f(x)-f(y)|, \qquad x,y\in[-\pi,\pi],$$

and $\omega(\delta, f; [a, b])$ for the local modulus of continuity on the interval $[a, b]$,

$$\omega(\delta, f; [a, b]) = \sup_{|x-y| \leq \delta} |f(x) - f(y)|, \qquad x, y \in [a, b]. \tag{1.12}$$

This paper is organized as follows. In Section 2 properties of second kind measures are studied, in particular, the absolute continuity of measure $d\tau$ on the interval $[a, b]$. Examples of Jacobi weight functions are examined. In Section 3 the asymptotic representation for second kind polynomials and functions is obtained. We should mention a recent paper [P], where similar problems are treated (cf. Remark 2 after Theorem 3.2 below).

2. Absolute Continuity of Second Kind Measures

In this section we present the proof of the main result concerning the absolute continuity of the second kind measure $d\tau$ on the interval $[a, b]$ in terms of the "first kind" measure $d\sigma$ (see Theorem 2.3 below).

We start with the following assertion, which is actually proved in [Ge1, Theorem 3.10] (cf. [MNT, Lemma 4.2]).

Lemma 2.1. *Let φ_n be the orthonormal polynomials with respect to measure $d\sigma$. For all points of continuity θ_1, θ_2 of the measure $d\sigma$*

$$\lim_{m \to \infty} \int_{\theta_1}^{\theta_2} |\varphi_m(\zeta)|^{-2} \, dt = \sigma(\theta_2) - \sigma(\theta_1) = \sigma\{[\theta_1, \theta_2]\}. \tag{2.1}$$

Lemma 2.2. *If for a sequence Λ of positive integers and for $\zeta = e^{it}$ we have $|\varphi_m(\zeta)| \leq L$ for $t \in [a_1, a_2]$ and $m \in \Lambda$, then*

$$|\psi_m(\zeta)| \geq (c_0 L)^{-1}, \qquad m \in \Lambda, \; t \in [a_1, a_2], \tag{2.2}$$

and $d\tau \in AC[a_1, a_2]$.

Proof. From the identity (1.8) with $z = \zeta$ it follows that

$$\operatorname{Re}\{\varphi_n(\zeta) \overline{\psi_n(\zeta)}\} = c_0^{-1},$$

and, hence, $|\varphi_n(\zeta) \psi_n(\zeta)| \geq c_0^{-1}$. For $m \in \Lambda$ and $t \in [a_1, a_2]$ we get

$$|\psi_m(\zeta)| \geq (c_0 |\varphi_m(\zeta)|)^{-1} \geq (c_0 L)^{-1}. \tag{2.3}$$

It is well known [ShTa, pp. 45–46] that in the mass point t_0 of the measure $d\tau$

$$\sum_{n=0}^{\infty} |\psi_n(\zeta_0)|^2 = 2\pi(\tau\{t_0\})^{-1}, \qquad \zeta_0 = e^{it_0}, \tag{2.4}$$

and, in particular, $\lim_{n\to\infty} \psi_n(\zeta_0) = 0$. The relation (2.3) means now that the measure $d\tau$ has no mass points on the interval $[a_1, a_2]$. Applying Lemma 2.1 to the measure $d\tau$, we obtain

$$\tau\{[t_1, t_2]\} \leqslant c_0 L(t_2 - t_1), \qquad a_1 \leqslant t_1 < t_2 \leqslant a_2,$$

what was to be proved. ∎

THEOREM 2.3. *Let φ be a density function of the measure $d\sigma$. If for some $p > 1$ we have $\varphi^{-1} \in L^p[a, b]$, $[a, b] \subset [-\pi, \pi]$, then $d\tau \in AC[a, b]$.*

Proof. Let $F(re^{i\theta}) = u(r, \theta) + v(r, \theta)$ (see (1.3)), so that for the "second kind" C-function G we have

$$\begin{aligned} g(r, \theta) \stackrel{\text{def}}{=} \operatorname{Re} G(re^{i\theta}) &= \frac{u(r, \theta)}{u^2(r, \theta) + v^2(r, \theta)} \leqslant \frac{1}{u(r, \theta)} \\ &= \left\{ \frac{1}{2\pi c_0} \int_{-\pi}^{\pi} P(r, t-\theta)\, d\sigma(t) \right\}^{-1} \\ &\leqslant \left\{ \frac{1}{2\pi c_0} \int_a^b P(r, t-\theta)\, \varphi(t)\, dt \right\}^{-1}. \end{aligned}$$

We assume further that $\theta \in [a, b]$. It can be readily shown (cf. [Z, Chap. 3, Theorem 6.18]) that for $0 \leqslant r < 1$ there exists a constant $K = K(a, b) > 0$, depending on a and b only, such that the relation

$$\frac{1}{2\pi} \int_a^b P(r, t-\theta)\, dt \geqslant K, \qquad a \leqslant \theta \leqslant b,$$

holds. By Schwarz' inequality we obtain

$$\begin{aligned} K^2 &\leqslant \left\{ \frac{1}{2\pi} \int_a^b P(r, t-\theta) \sqrt{\varphi(t)}\, \frac{dt}{\sqrt{\varphi(t)}} \right\}^2 \\ &\leqslant \left\{ \frac{1}{2\pi} \int_a^b P(r, t-\theta)\, \varphi(t)\, dt \right\} \left\{ \frac{1}{2\pi} \int_a^b P(r, t-\theta)\, \frac{dt}{\varphi(t)} \right\}, \end{aligned}$$

whence it follows that

$$0 \leqslant \frac{K^2}{c_0} g(r, \theta) \leqslant \frac{1}{2\pi} \int_a^b P(r, t-\theta) \frac{dt}{\varphi(t)}, \qquad \theta \in [a, b].$$

Let us define the 2π-periodic functions

$$\varphi_1(t) = \begin{cases} \varphi^{-1}(t), & a \leqslant t \leqslant b, \\ 0, & t \notin [a, b], \end{cases} \qquad g_1(r, \theta) = \begin{cases} g(r, \theta), & a \leqslant \theta \leqslant b, \\ 0, & \theta \notin [a, b], \end{cases}$$

for t, $\theta \in [-\pi, \pi]$. Then

$$0 \leqslant \frac{K^2}{c_0} g(r, \theta) \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} P(r, t-\theta)\, \varphi_1(t)\, dt \overset{\text{def}}{=} h(r, \theta).$$

The function $\varphi_1 \in L^p$ and

$$\|\varphi_1\|_p = \left\{ \frac{1}{2\pi} \int_a^b |\varphi^{-1}(t)|^p\, dt \right\}^{1/p}, \qquad \|g_1\|_p = \left\{ \frac{1}{2\pi} \int_a^b |g(r, \theta)|^p\, d\theta \right\}^{1/p}. \quad (2.5)$$

By W. Young's inequality [Z, Chap. 2, Theorem 1.15], $g_1 \in L^p$ and

$$\frac{K^2}{c_0} \|g_1\|_p \leqslant \|h\|_p \leqslant \|P\|_1\, \|\varphi_1\|_p.$$

The latter relation along with (2.5) means that the set of functions $\{g(r, \cdot)\}_{r \geqslant 0}$ is bounded in $L^p[a, b]$.

Choose a subsequence $g\{(r_k, \cdot)\}$, converging in $L^q[a, b]$, $q^{-1} = 1 - p^{-1}$, when $r_k \nearrow 1$. Since $\lim_{r \to 1-0} g(r, \theta) = c_0^{-1} \tau'(\theta)$ a.e., then for every $a \leqslant \theta_1 < \theta_2 < b$ we have

$$\lim_{k \to \infty} \int_{\theta_1}^{\theta_2} g(r_k, \theta)\, d\theta = c_0^{-1} \int_{\theta_1}^{\theta_2} \tau'(\theta)\, d\theta.$$

Absolute continuity of the measure $d\tau$ can be deduced now from the inversion formula (1.5). Thus Theorem 2.3 has been proved. ▮

Applying Theorem 2.3 we shall study the second kind measures $d\tau$, corresponding to the generalized Jacobi weight functions w on the unit circle

$$d\sigma(t) = w(t)\, dt = h(t) \prod_{\nu=1}^{N} |\zeta - \zeta_\nu|^{2\gamma_\nu}\, dt, \qquad \zeta_\nu = e^{it_\nu}, \quad (2.6)$$

where

$$-\pi < t_1 < \cdots < t_N \leqslant \pi, \qquad \gamma_1, ..., \gamma_N > -\tfrac{1}{2}, \quad (2.7)$$

and with respect to the "regular" factor h we assume that

$$h \in C_{2\pi}, \qquad h(t) > 0, \qquad \omega(t, h)\, t^{-1} \in L^1(0, \pi). \tag{2.8}$$

THEOREM 2.4. *Let w be the generalized Jacobi weight function* (2.6)–(2.8) *and $J = \{t_j : \gamma_j > \frac{1}{2}\}$. Then $d\tau \in AC[a, b]$ for each interval $[a, b]$, containing no points from J.*

Proof. Let $\gamma = \max\{\gamma_j : t_j \in [a, b]\}$. The case $\gamma < \frac{1}{2}$ is contained in Theorem 2.3. The assumption (2.8) can be replaced here by $h^{\pm 1} \in L^\infty$ (cf. [N, pp. 35, 48–49] for the interval $[-1, 1]$). The case $\gamma = \frac{1}{2}$ requires more subtle considerations.

As was shown in the proof of Lemma 2.2 (see (2.3)), we have $|\psi_n(\zeta)| \geqslant |c_0 \varphi_n(\zeta)|^{-1}$. Under the conditions (2.6)–(2.8) the inequality

$$|\varphi_n(\zeta)| \leqslant A_1 \prod_{\nu=1}^{N} \left(\left| \sin \frac{t - t_\nu}{2} \right| + \frac{1}{n} \right)^{-\gamma_\nu} \tag{2.9}$$

holds (cf. [B1, Proposition 2.2]). Hence, it follows that

$$|\psi_n(\zeta)| \geqslant A_2 \prod_{\nu=1}^{N} \left(\left| \sin \frac{t - t_\nu}{2} \right| + \frac{1}{n} \right)^{\gamma_\nu}. \tag{2.10}$$

It is easy to check that the measure $d\tau$ is continuous (the mass points are absent) in the interval $[a, b]$. Indeed, if $t \neq t_\nu$, $1 \leqslant \nu \leqslant N$, then $|\psi_n(\zeta)| \geqslant A_3 > 0$, which leads to $\tau(\{t\}) = 0$ (see (2.4)). For $t = t_\nu \in [a, b]$ we have $|\psi_n(\zeta_\nu)| \geqslant A_4 n^{-\gamma_\nu}$, and, since $\gamma = \frac{1}{2}$ now,

$$\sum_{n=0}^{\infty} |\psi_n(\zeta_\nu)|^2 = \infty,$$

which again leads to $\tau(\{t_\nu\}) = 0$.

We use the limit relation (cf. [Sz], [N, p. 70])

$$\lim_{n \to \infty} \int_{-\pi}^{\pi} |\psi_n^*(\zeta) - D^{-1}(\zeta, \tau)|^2 \, d\tau(t) = 0.$$

Since

$$\int_{-\pi}^{\pi} |\psi_n^*(\zeta) - D^{-1}(\zeta, \tau)|^2 \, d\tau(t)$$
$$= \int_{-\pi}^{\pi} |\psi_n^*(\zeta)\, D(\zeta, \tau) - 1|^2 \, dt + \int_{-\pi}^{\pi} |\psi_n^*(\zeta) - D^{-1}(\zeta, \tau)|^2 \, d\tau_s(t),$$

we obtain

$$\lim_{n\to\infty}\int_{-\pi}^{\pi}|\psi_n^*(\zeta)\,D(\zeta,\tau)-1|^2\,dt=0.$$

Thus there exists a sequence $\Lambda=\{0\leqslant n_1<n_2<\cdots\}$ such that the equality

$$\lim_{m\in\Lambda}|\psi_m^*(\zeta)|^{-2}=|D(\zeta,\tau)|^2=\tau'(t) \tag{2.11}$$

holds a.e. By Lemma 2.1,

$$\lim_{m\in\Lambda}\int_{u_1}^{u_2}|\psi_m(\zeta)|^{-2}\,dt=\tau\{[u_1,u_2]\} \tag{2.12}$$

for all u_1, $u_2\in[a,b]$. Let $t_\nu<u_1<u_2<t_{\nu+1}$. By Lebesgue's Bounded Convergence Theorem (see (2.10)) we obtain from (2.11), (2.12)

$$\lim_{m\in\Lambda}\int_{u_1}^{u_2}|\psi_m(\zeta)|^{-2}\,dt=\int_{u_1}^{u_2}\tau'(t)\,dt=\tau\{[u_1,u_2]\}.$$

Letting $u_1\searrow t_\nu$ and $u_2\nearrow t_{\nu+1}$ in the latter equality, we conclude that

$$\int_{t_\nu}^{t_{\nu+1}}\tau'(t)\,dt=\tau\{[t_\nu,t_{\nu+1}]\},\qquad \nu=1,2,...,N-1,$$

and, therefore,

$$\int_{u_1}^{u_2}\tau'(t)\,dt=\tau\{[u_1,u_2]\}$$

for every u_1, $u_2\in[a,b]$. Thus, Theorem 2.4 has been proved. ∎

COROLLARY. *Let w be the weight function* (2.6)–(2.8). *If*

$$\max_{1\leqslant j\leqslant N}\gamma_j\leqslant\tfrac{1}{2},$$

then the second kind measure $d\tau\in AC$.

EXAMPLE. Consider the Jacobi weight function

$$w(t)=|e^{it}-1|^{2\gamma_1}\,|e^{it}+1|^{2\gamma_2}=2^{\gamma_1+\gamma_2}(1-\cos t)^{\gamma_1}\,(1+\cos t)^{\gamma_2},$$

$$c_0=(2\pi)^{-1}\int_{-\pi}^{\pi}w(t)\,dt=\frac{4^{\gamma_1+\gamma_2}}{\pi}\,B\left(\gamma_1+\frac{1}{2},\gamma_2+\frac{1}{2}\right),$$

where $B(x, y)$ is the Euler beta function. In this situation the corresponding C-function $F(z, w)$ can be expressed by means of the hypergeometric function (cf. [L, Chap. 6])

$$F(a, b; c; z) = 1 + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \qquad (x)_n = x(x+1) \cdots (x+n-1).$$

In fact,

$$2\pi c_0 F(z, w) = \int_{-\pi}^{\pi} P(r, \theta - t)\, w(t)\, dt + i \int_{-\pi}^{\pi} Q(r, \theta - t)\, w(t)\, dt,$$

and, since the weight function is even now, we have for $z = r$, $0 < r < 1$, and $0 < \rho = 4r(1+r)^{-2} < 1$

$$\begin{aligned} \pi c_0 F(r, w) &= \int_0^{\pi} P(r, t)\, w(t)\, dt \\ &= 2^{\gamma_1 + \gamma_2}(1 - r^2) \int_0^{\pi} \frac{(1 - \cos t)^{\gamma_1} (1 + \cos t)^{\gamma_2}}{1 - 2r \cos t + r^2}\, dt \\ &= 4^{\gamma_1 + \gamma_2}(1 - r^2) \int_0^1 \frac{(1-x)^{\gamma_1 - 1/2}\, x^{\gamma_2 - 1/2}}{(1+r)^2 - 4rx}\, dx \\ &= 4^{\gamma_1 + \gamma_2} \frac{1-r}{1+r} \int_0^1 \frac{(1-x)^{\gamma_1 - 1/2}\, x^{\gamma_2 - 1/2}}{1 - \rho x}\, dx. \end{aligned}$$

The Euler integral for the hypergeometric function (cf. [L, Chap. 6.2.3, formula (1)]) gives

$$\begin{aligned} &\int_0^1 \frac{(1-x)^{\gamma_1 - 1/2}\, x^{\gamma_2 - 1/2}}{1 - \rho x}\, dx \\ &\quad = B\left(\gamma_1 + \frac{1}{2}, \gamma_2 + \frac{1}{2}\right) F\left(1, \gamma_2 + \frac{1}{2}; \gamma_1 + \gamma_2 + 1; \rho\right). \end{aligned}$$

Consequently, by the uniqueness theorem for analytic functions, we get

$$F(z, w) = \frac{1-z}{1+z} F\left(1, \gamma_2 + \frac{1}{2}; \gamma_1 + \gamma_2 + 1; \frac{4z}{(1+z)^2}\right),$$

and for the second kind C-function $G(z, w)$ (1.9)

$$G(z, w) = \frac{1+z}{1-z} F^{-1}\left(1, \gamma_2 + \frac{1}{2}; \gamma_1 + \gamma_2 + 1; \frac{4z}{(1+z)^2}\right),$$

respectively. For $\gamma_1 > \frac{1}{2}$ we have

$$c - b - a = (\gamma_1 + \gamma_2 + 1) - (\gamma_2 + \tfrac{1}{2}) = \gamma_1 - \tfrac{1}{2} > 0$$

so that, as is well known (cf. [L, Chap. 6.8, formula (1)]),

$$\lim_{x \to 1-0} F\left(1, \gamma_2 + \frac{1}{2}; \gamma_1 + \gamma_2 + 1; \frac{4x}{(1+x)^2}\right) = \frac{\gamma_1 + \gamma_2}{\gamma_1 - \frac{1}{2}}.$$

Hence

$$\lim_{x \to 1-0} G(x, w)(1-x) = 2\,\frac{\gamma_1 - \frac{1}{2}}{\gamma_1 + \gamma_2} > 0,$$

i.e. (see (1.6)), $\tau(\{0\}) > 0$.

In the special case $\gamma_1 = \gamma_2 = \gamma$ the quadratic transform for the hypergeometric function (cf. [L, Chap. 6.7, formula (3)]) leads to the following expression for the C-function $F(z, w_\gamma)$, corresponding to the weight function $w_\gamma(t) = 4^\gamma\,|\sin t|^{2\gamma}$:

$$F(z, w_\gamma) = (1 - z^2)\, F(1, 1 - \gamma; 1 + \gamma; z^2).$$

If $\gamma = \frac{1}{2}$, then (cf. [L, Chap. 3.1.1, formula (7)])

$$F(z, w_\gamma) = (1 - z^2) F\left(1, \frac{1}{2}; \frac{3}{2}; z^2\right) = \frac{1 - z^2}{2z} \log \frac{1+z}{1-z}.$$

By Theorem 2.4, the second kind measure $d\tau$ is absolutely continuous and the corresponding density function is equal to

$$\psi(\theta) = \frac{2}{|\sin \theta|} \left\{ \log^2 \left|\cot \frac{\theta}{2}\right| + \frac{\pi^2}{4} \right\}^{-1}.$$

For $\gamma = 1$ we have $F(z, w_1) = 1 - z^2$, $G(z, w_1) = (1 - z^2)^{-1}$, and the second kind measure $d\tau$ has two mass points with $\tau(\{0\}) = \tau(\{-\pi\}) = \pi$.

3. Asymptotic Representation for Second Kind Polynomials and Functions

Under the asymptotic representation for the orthogonal polynomials we mean the limit relation

$$\lim_{n \to \infty} \psi_n^*(e^{i\theta}) = D^{-1}(e^{i\theta}, \tau)$$

uniformly on a closed set $E \subset I = [-\pi, \pi]$ (cf., e.g., [Go2, p. 147]). Here we consider the asymptotic representation for the second kind polynomials and $E = [a, b]$.

The assertion we begin with may be regarded as a local analog of Zygmund's inequality (cf. [Z, Chap. 3, Theorem 13.30]) for conjugate functions.

LEMMA 3.1. *Let $[a, b] \subset I = [-\pi, \pi]$. Let the measure $d\sigma \in AC[a, b]$ with the density function $\varphi = \sigma'$, continuous on $[a, b]$. Assume that for the local modulus of continuity $\omega(t, \varphi; [a, b])$ (1.12) the inequality*

$$\omega(t, \varphi; [a, b]) \leqslant \omega_0(t) \tag{3.1}$$

holds, where the continuous function ω_0 satisfies the Dini condition

$$\int_0^{t_0} \frac{\omega_0(t)}{t}\, dt < \infty, \qquad 0 < t_0 < \min(1, b-a). \tag{3.2}$$

Then the conjugate function $\tilde{\varphi}_\sigma$ (1.11) is continuous in the open interval (a, b) and for each interval $[a_1, b_1] \subset (a, b)$ there is a positive constant $K = K(\sigma, a_1, b_1)$ such that

$$\omega(t, \tilde{\varphi}_\sigma; [a_1, b_1]) \leqslant K \left\{ \int_0^t \frac{\omega_0(u)}{u}\, du + t \int_t^{t_0} \frac{\omega_0(u)}{u^2}\, du \right\}$$
$$= K \int_0^t du \int_u^{t_0} \frac{\omega_0(v)}{v^2}\, dv.$$

Proof. Consider the 2π-periodic functions

$$f(\theta) = \begin{cases} \varphi(\theta), & \theta \in [a, b], \\ 0, & \theta \notin [a, b], \end{cases} \qquad g(\theta) = \begin{cases} \varphi(\theta) & \theta \in [a, b], \\ g_0(\theta), & \theta \notin [a, b], \end{cases} \qquad \theta \in I.$$

The function g_0 here is a linear function, such that $g \in C_{2\pi}$. It is easy to check that

$$\omega(t, g) = O(\omega(t, \varphi; [a, b])). \tag{3.4}$$

We assume further that $\theta \in [a_1, b_1]$. Then, putting $I' = I \backslash [a, b]$, we have a.e. (see (1.11))

$$c_0 \tilde{\varphi}_\sigma(\theta) = \lim_{r \to 1-0} \frac{1}{2\pi} \int_{-\pi}^{\pi} Q(r, \theta - t)\, d\sigma(t)$$
$$= \lim_{r \to 1-0} \left\{ \frac{1}{2\pi} \int_a^b Q(r, \theta - t)\, \varphi(t)\, dt + \frac{1}{2\pi} \int_{I'} Q(r, \theta - t)\, d\sigma(t) \right\}$$
$$= \tilde{f}(\theta) + \frac{1}{2\pi} \int_{I'} \cot \frac{\theta - t}{2}\, d\sigma(t) = \tilde{f}(\theta) + f_1(\theta),$$

where $\tilde{f}$ is an ordinary conjugate function to f, and

$$\omega(t, f_1; [a_1, b_1]) = O(t). \tag{3.5}$$

Similarly, we get

$$\tilde{g}(\theta) = \tilde{f}(\theta) + g_1(\theta)$$

with the same property (3.5) for $\omega(t, g_1; [a_1, b_1])$. Therefore,

$$c_0 \tilde{\varphi}_\sigma(\theta) = \tilde{g}(\theta) + f_1(\theta) - g_1(\theta), \qquad \theta \in [a_1, b_1], \tag{3.6}$$

and by (3.5), (3.4), (3.2), $\tilde{\varphi}_\sigma$ is continuous on (a, b) (cf. [Ga, Chap. 3, Theorem 1.3]). Moreover,

$$\omega(t, \tilde{\varphi}_\sigma; [a_1, b_1]) = O(\omega(t, \tilde{g}; [a_1, b_1])) = O(\omega(t, \tilde{g})). \tag{3.7}$$

Applying Zygmund's inequality to the function $\tilde{g}$ (see (3.1), (3.4), (3.2)) and considering (3.7), we reach the inequality (3.3). ∎

THEOREM 3.2. *Let the measure $d\sigma$ belong to the Szegő class, $d\sigma \in AC(a, b)$. Let the density function φ be positive and continuous in the open interval (a, b) and let*

$$\int_0^{t_0} \frac{\omega(t, \varphi; [a_1, b_1])}{t} \log \frac{1}{t}\, dt < \infty, \qquad 0 < t_0 < \min(1, b_1 - a_1), \tag{3.8}$$

hold for each interval $[a_1, b_1] \subset (a, b)$. Then the second kind measure $d\tau$ belongs to the Szegő class, $d\tau \in AC(a, b)$, the second kind density function $\psi(t)$ is positive and continuous on (a, b), and

$$\int_0^{t_0} \frac{\omega(t, \psi; [a_1, b_1])}{t}\, dt < \infty. \tag{3.9}$$

The asymptotic representation

$$\lim_{n \to \infty} \psi_n^*(e^{i\theta}) = D^{-1}(e^{i\theta}, \tau) \tag{3.10}$$

holds uniformly inside (a, b).

Proof. As was mentioned in Section 1, the measures $d\sigma$ and $d\tau$ belong to the Szegő class simultaneously. According to (1.10), a.e. on $[-\pi, \pi]$

$$\psi(t) = c_0^2 \frac{\varphi(t)}{\varphi^2(t) + \tilde{\varphi}_\sigma^2(t)}. \tag{3.11}$$

Hence, by Lemma 3.1, ψ is positive and continuous on (a, b). The conclusion $d\tau \in AC(a, b)$ is now an immediate consequence of Theorem 2.3.

Next we turn to the relation (3.9). It can be readily shown from (3.11) that

$$\omega(t, \psi; [a_1, b_1]) = O\{\omega(t, \varphi; [a_1, b_1]) + \omega(t, \tilde{\varphi}_\sigma; [a_1, b_1])\}. \quad (3.12)$$

In order to establish (3.9) we need to estimate the second term in the right-hand side of (3.12). Because of the assumption (3.8) we can apply the inequality (3.3),

$$\int_0^{t_0} \frac{\omega(x, \tilde{\varphi}_\sigma; [a_1, b_1])}{x} \, dx \leqslant K \int_0^{t_0} \frac{dx}{x} \int_0^x dt \int_t^{t_0} \frac{\omega(u, \varphi; [a_1, b_1]}{u^2} \, du$$

$$= \int_0^{t_0} \frac{\omega(u, \varphi; [a_1, b_1])}{u^2} \int_0^u \log \frac{t_0}{t} \, dt \, du < \infty,$$

which leads to (3.9).

V. M. Badkov proved (cf. [B2, Theorem 2]) that the asymptotic representation (3.10) holds uniformly inside (a, b) provided the measure $d\tau$ satisfies (S), $d\tau \in AC(a, b)$, $\psi(t) = \tau'(t)$ is a positive and continuous function on (a, b), and for each $[a_1, b_1] \subset (a, b)$ the local modulus of continuity $\omega(t, \psi; [a_1, b_1])$ satisfies the Dini condition (3.2). Thereby the proof of Theorem 3.2 is completed. ∎

Remark 1. If the measure $d\sigma \in AC$, the density function φ is positive and continuous in $[-\pi, \pi]$, and

$$\int_0^1 \frac{\omega(t, \varphi)}{t} \log \frac{1}{t} \, dt < \infty,$$

then the asymptotic representation (3.10) holds uniformly on the whole circle.

Remark 2. In [P] the asymptotic behavior for the second kind functions

$$\tilde{g}_n(z, d\sigma) = \frac{\varphi_n(z) \, F(z) + \psi_n(z)}{z^n}, \qquad \tilde{h}_n(z, d\sigma) = \frac{\varphi_n^*(z) \, F(z) - \psi_n^*(z)}{z^{n+1}}$$

is studied. Theorem 3.2 gives more general conditions (cf. [P, Remark 2.1(b)]) for the relations (cf. [P, formula (2.19)])

$$\lim_{n \to \infty} \tilde{g}_n(\zeta, d\sigma) = \frac{2}{c_0} D(\zeta, d\sigma), \qquad \lim_{n \to \infty} \tilde{h}_n(\zeta, d\sigma) = 0$$

to hold uniformly inside (a, b).

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